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THE QUASI-STATIONARY MODE OF A HELMHOLTZ RESONATOR†

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Complete asymptotic forms of the poles of Green's functions of a Helmholtz resonator are constructed with respect to a small parameter (the linear dimensions of the aperture), which converge to the zeroth eigenvalue of the "closed" resonator. The principal terms of the asymptotic forms of the solutions of the corresponding scattering and radiation problems are obtained.

THE ACOUSTIC Helmholtz resonator is a surface obtained from the boundary of a finite volume with a small opening cut in it [1–3]. It has been shown [4, 5] that if $k_0^2 \neq 0$ is the eigenvalue of the unperturbed problem, then for certain frequencies k close to k_0 when a plane wave is incident the field reflected from the Helmholtz resonator is different from the field scattered by the "closed" resonator by a quantity of the order of unity when $\varepsilon \to 0$ ($0<\varepsilon^2 \ll 1$ is the linear dimension of the aperture). In the quasi-stationary case ($k_0=0$) the situation is quite different: the field reflected from the Helmholtz resonator differs, under peak conditions, from the field scattered by the "closed" resonator by a quantity $O(\varepsilon^{-1})$.

This difference is explained as follows. If k_0^2 is the simple eigenvalue of the "closed" resonator (and the minimum eigenvalue $k_0^2=0$ is in fact this), Green's function of the unperturbed internal problem for k close to k_0 can be represented in the form

$$G^{\text{in}}(x,y,k)=(k_0^2-k^2)^{-1}\psi(x)\psi(y)+\tilde{G}(x,y,k) \tag{0.1}$$

where $\tilde{G}(x, y, k)$ is a function that is regular in a certain neighbourhood of k_0, ψ is the corresponding eigenfunction, orthonormalized in $L_2(\Omega)$, and Ω is the interior of the resonator. If $k_0 \neq 0$, then, as can be seen from (0.1), this value is a first-order pole of the function $G^{\text{in}}(x, y, k)$. In this case, Green's function $G_\varepsilon(x, y, k)$ of the Helmholtz resonator has a unique first-order pole τ_ε, which approaches k_0 as $\varepsilon \to 0$, and for k close to k_0 the following representation holds [6]

$$G_\varepsilon(x,y,k)=(\tau_\varepsilon^2-k^2)^{-1}\Psi_\varepsilon(x)\Psi_\varepsilon(y)+\tilde{G}_\varepsilon(x,y,k) \tag{0.2}$$

where, as $\varepsilon \to 0$, the quasi-eigenfunction $\Psi_\varepsilon \to \psi$ in $W_2^1(\Omega)$, $\Psi_\varepsilon \to 0$ in $W_2^1(K \setminus \overline{\Omega})$, and K is any compactum in $\mathbf{R}^3$. Everywhere henceforth the convergence of the functions is understood to be in these norms.

The value of $k_0=0$ is a second-order pole with respect to the k functions $G^{\text{in}}(x, y, k)$. A consequence of this is the existence in Green's function of the Helmholtz resonator of two first-order poles $\tau_\varepsilon^{(j)} \to 0$ connected by the relation $\tau_\varepsilon=\tau_\varepsilon^{(1)}=-\bar{\tau}_\varepsilon^{(2)}$

$$G_\varepsilon(x,y,k)=(2\operatorname{Re}\tau_\varepsilon)^{-1}((\tau_\varepsilon-k)^{-1}\Psi_\varepsilon(x)\Psi_\varepsilon(y)+(\bar{\tau}_\varepsilon+k)^{-1}\overline{\Psi}_\varepsilon(x)\overline{\Psi}_\varepsilon(y))+\tilde{G}_\varepsilon(x,y,k) \tag{0.3}$$

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where $\Psi_\varepsilon \to \psi \equiv \operatorname{mes}^{-1/2}\Omega$ inside Ω and $\Psi_\varepsilon \to 0$ outside Ω [6, 7].

The first term on the right-hand side of (0.2), like the first two terms on the right-hand of (0.3), are the terms that give rise to resonance phenomena. The presence in (0.3), are the terms that give rise to resonance phenomena. The presence in (0.3) of the factor $(2\operatorname{Re}\tau_\varepsilon)^{-1}$ implies that the quasi-stationary mode differs from those in the case when $k_0 \neq 0$. Since $\tau_\varepsilon \to 0$, $\operatorname{Im}\tau_\varepsilon \to 0$, $\Psi_\varepsilon \to 0$, outside the resonator as $\varepsilon \to 0$, to obtain the principal terms of the asymptotic forms of the corresponding boundary-value problems it is necessary to know the asymptotic forms of τ_ε and Ψ_ε. We will construct these expansions below.

1. FORMULATION OF THE PROBLEM AND FUNDAMENTAL PROPOSITIONS

Suppose a limited region $\Omega \subset \mathbf{R}^3$ has a fairly smooth boundary Γ_0, and Γ_ε is obtained from Γ_0 by cutting out an aperture ω_ε with linear dimensions $O(\varepsilon^2)$ (an acoustic Helmholtz resonator). If the space is filled with a homogeneous and isotropic liquid or gaseous medium, the potential u_ε of its velocity $\mathbf{v}_\varepsilon = \operatorname{grad} u_\varepsilon$ is a solution of the following boundary-value problem

$$(\Delta + k^2)u_\varepsilon = 0,\ x \in \mathbf{R}^3 \setminus \overline{\Gamma}_\varepsilon,\ \partial u_\varepsilon / \partial \mathbf{n} = f,\ x \in \Gamma_\varepsilon \tag{1.1}$$

$$\partial u_\varepsilon / \partial r - iku_\varepsilon = o(r^{-1}),\ r \to \infty \tag{1.2}$$

which satisfy the Meixner condition on the edge of the surface Γ_ε. Here and everywhere henceforth $\mathbf{n}$ is the outward normal to Ω and $x = (x_1, x_2, x_3)$, $r = |x|$. The function f is arbitrary, the surface Γ_ε can be regarded as a radiating surface. The problem of finding the scattered field u_ε which occurs when an external field u^{out} is reflected from an ideal rigid surface Γ_ε reduces to the solution of the boundary-value problem (1.1) and (1.2). In this case, we must put $f = -\partial u^{\text{out}} / \partial \mathbf{n}$ in (1.1).

Suppose $u_0(x, k) = U^{\text{ex}}(x, k)$ is the solution of the Neumann problem for the Helmholtz operator outside Ω, and $U^{\text{in}}(x, k)$ is the regular part of the solution of the Neumann problem in Ω. The expression for the solution of boundary-value problem (1.1), (1.2) in terms of Green's function (0.3) gives the following representation for it [6, 7]

$$u_\varepsilon(x,k) = \frac{1}{2\operatorname{Re}\tau_\varepsilon}\left(\frac{\Psi_\varepsilon(x)}{\tau_\varepsilon - k}\int_{\Gamma_0}\{\Psi_\varepsilon\} f ds + \frac{\overline{\Psi}_\varepsilon(x)}{\overline{\tau}_\varepsilon + k}\int_{\Gamma_0}\{\overline{\Psi}_\varepsilon\} f ds\right) + U_\varepsilon(x,k) \tag{1.3}$$

$$\{\Psi_\varepsilon(x)\} = \lim_{y \to x}\Psi_\varepsilon(y) - \lim_{z \to x}\Psi_\varepsilon(z)\quad (x \in \Gamma_0,\ y \in \Omega,\ z \notin \overline{\Omega}) \tag{1.4}$$

$$\int_{\Gamma_0}\{\Psi_\varepsilon\} f ds \to \psi a_f = \psi \int_{\Gamma_0} f ds$$

where $U_\varepsilon \to U^{\text{ex,in}}$ as $\varepsilon \to 0$ uniformly with respect to k, and the quasi-eigenfunction $\Psi_\varepsilon(x)$ is the solution of boundary-value problem (1.1) when $k = \tau_\varepsilon$.

Assuming that Ω in the neighbourhood of the origin of coordinates coincides with the half-space $x_3 > 0$, ω is a two-dimensional region with a smooth boundary in the $x_3 = 0$ plane, and $\omega_\varepsilon = \{x : x\varepsilon^{-2} \in \omega\}$, in the following section we will show that

$$\tau_\varepsilon = \sum_{j=1}^{\infty}\varepsilon^j\tau_j,\quad \tau_1 = \psi(\pi c_\omega)^{1/2},\quad \tau_2 = 0 \tag{1.5}$$

$$\tau_3 = -\frac{1}{2}\psi(\pi c_\omega)^{3/2}(g_{00}^{\text{in}}(0,0) + g_{00}^{\text{ex}}(0.0)),\quad \operatorname{Im}\tau_4 = -\frac{1}{2}(\pi\psi c_\omega)^2\sigma$$

where c_ω is the capacity of the "plate" ω [8, 9], is the transverse cross-section [2, 10] of Green's function $G^{\text{ex}}(x, y, 0)$ of the Neumann problem for the Laplace operator outside Ω when $y = 0$, and $g_{00}^{\text{in,ex}}(x, k)$ are the regular parts of the functions $G^{\text{in,ex}}(x, 0, k)$. Since, in addition to the pole $\tau_\varepsilon^{(1)} = \tau_\varepsilon$ there is also the pole $\tau_\varepsilon^{(2)} = -\bar{\tau}_\varepsilon$, the sign of $\operatorname{Re}\tau_\varepsilon > 0$ is chosen in order to be more specific.

The following expansion holds for the quasi-eigenfunction $\psi_\varepsilon(x)$

$$
\begin{gathered}
\psi_\varepsilon(x) = -k^2 \sum_{j=0}^{\infty} \varepsilon^j R_{[j/2]}^{(j)}(D_y) G^{\text{in}}(x,0,k), \quad x \in \Omega \setminus S(\varepsilon), \quad R_0^{(0)} = \psi^{-1} \\
\psi_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j \upsilon_j\left(\frac{x}{\varepsilon^2}\right), \quad x \in S(2\varepsilon), \quad \upsilon_0(\xi) = \psi Y(\xi), \quad \upsilon_1(\xi) = 0 \\
\psi_\varepsilon(x) = k^2 \sum_{j=0}^{\infty} \varepsilon^j R_{[j/2]}^{(j)}(D_y) G^{\text{ex}}(x,0,k), \quad x \notin \Omega \cup S(\varepsilon) \\
Y(\xi) = \begin{cases} 1 - \dfrac{1}{2}\tilde{Y}(\xi), & \xi_3 > 0 \\ \dfrac{1}{2}\tilde{Y}(\xi), & \xi_3 < 0 \end{cases}
\end{gathered}
\tag{1.6}
$$

where $k = \tau_\varepsilon$, $S(t)$ is a sphere of radius t with centre at the origin of coordinates, $R_j^{(q)}(D_y)$ are differential polynomials of the jth order with respect to the variable y with constant coefficients, and $\tilde{Y}(\xi) \in W_{2,\text{loc}}^1(\mathbf{R}^3)$ is a function that is harmonic outside $\bar{\omega}$ and which falls off at infinity and is equal to unity on ω. Note that the quasi-eigenfunctions $\psi_\varepsilon(x)$ and $\Psi_\varepsilon(x)$ which occur in (1.6) and in (0.3) and (1.3) are equal, apart from the scalar factor $1 + o(1)$.

It can be seen from (1.3) that resonance phenomena will be observed to the greatest extent for real values of $k = k(\varepsilon)$ in the peak modes $k = \tau_\varepsilon + O(\operatorname{Im}\tau_\varepsilon)$, $k = -\bar{\tau}_\varepsilon + O(\operatorname{Im}\tau_\varepsilon)$. Without loss of generality we will henceforth only consider the first of these, i.e. we will assume that the real $k = k(\varepsilon)$ has the form

$$
k = \varepsilon\tau_1 + \varepsilon^3\tau_3 + \varepsilon^4(k_4 + o(1))
\tag{1.7}
$$

For a radiating surface Γ_ε, in the case of the general state, $a_f \neq 0$. Substituting the asymptotic expansions (1.5) and (1.6) into (1.3) and taking (1.4) into account we obtain the following representation for the solution of boundary-value problem (0.1), (0.2)

$$
\begin{gathered}
u_\varepsilon(x;k) \sim \varepsilon^{-5} A_f, \quad x \in \Omega \setminus S(\varepsilon); \quad u_\varepsilon(x;k) \sim \varepsilon^{-5} A_f Y(\xi), \quad x \in S(2\varepsilon) \\
u_\varepsilon(x;k) \sim A_f \varepsilon^{-3} \pi c_\omega G^{\text{ex}}(x,0,k), \quad x \notin \Omega \cup S(\varepsilon) \\
A_f = \tfrac{1}{2} a_f \psi(\tau_4 - k_4)^{-1} (\pi c_\omega)^{-1/2}, \quad \xi = x/\varepsilon^2
\end{gathered}
\tag{1.8}
$$

When finding the scattered field u_ε that occurs when u^{out} is diffracted by an ideally rigid surface Γ_ε, the situation is somewhat different since in this case $a_f = 0$. Suppose $u_0(x;\, k)$ is the scattered field which occurs outside Ω when the external field $u^{\text{out}}(x;\, k)$ is reflected from Γ_0 (the solution of the Neumann problem outside Ω when $f = -\partial u^{\text{out}}/\partial \mathbf{n}$), and $u = u_0 + u^{\text{out}}$ in $\mathbf{R}^3 \setminus \Omega$. It can be shown [5, 6], that in the peak mode (1.7)

$$
\int_{\Gamma_0} \{\psi_\varepsilon(x)\} \frac{\partial u^{\text{out}}(x;k)}{\partial \mathbf{n}}\, ds \sim -\varepsilon^2 \tau_1^2 R_0^{(0)} u(0,0)
\tag{1.9}
$$

Substituting (1.5), (1.6) and (1.9) into (1.3) we obtain the principal terms of the asymptotic forms for the scattered field

$$
u_\varepsilon(x;k) \sim \varepsilon^{-3} b, \quad x \in \Omega \setminus S(\varepsilon); \quad u_\varepsilon(x;k) \sim \varepsilon^{-3} b Y(\xi), \quad x \in S(2\varepsilon)
$$

$$u_\varepsilon(x;k) \sim \varepsilon^{-1} b\pi c_\omega G^{ex}(x,0,k), \quad x \notin \Omega \cup S(\varepsilon) \tag{1.10}$$

$$b = \tfrac{1}{2}\psi(\pi c_\omega)^{1/2}(\tau_4 - k_4)^{-1} u(0,0), \quad \xi = x/\varepsilon^2$$

2. CONSTRUCTION OF THE ASYMPTOTIC FORMS OF THE POLES AND QUASI-EIGENFUNCTIONS

We will show that relations (1.5) and (1.6) hold. The series (1.6) are constructed by the method of matched asymptotic expansions. The boundary-value problems for the second-order coefficients of (1.6) are obtained as follows [5, 11]. We substitute this series and also series (1.5) into (1.1) with $f=0$, change to the variable $\xi = x\varepsilon^{-2}$ in the equation and boundary conditions, write the equation separately for the same powers of ε, and pass to the formal limit as $\varepsilon \to 0$. We obtain

$$\Delta_\xi \upsilon_j = -\sum_{i=2}^{j-4} \lambda_j \upsilon_{j-i-4}, \quad \xi \notin \bar{\gamma}, \quad \partial\upsilon_j / \partial\xi_3 = 0, \quad \xi \in \gamma, \quad \gamma = \mathbf{R}^2 \setminus \bar{\omega} \tag{2.1}$$

where $\mathbf{R}^2$ is the $\xi_3 = 0$ plane and λ_i are the coefficients of the series τ_ε^2.

When $t \geq 1$, the differential polynomials will be sought in the form

$$R_j^{(t)}(D_y) = \sum_{i=1}^{j} P_i^{(t)}(D_y), \quad P_i^{(t)}(D_y) = \sum_{q=0}^{i} a_{qi}^{(t)} \frac{\partial^i}{\partial^q y_1 \partial^{i-q} y_2}$$

where $a_{ji}^{(q)}$ are certain constants. We will denote the first and third series of (1.6) by $\psi_\varepsilon^{in}(x, k)$ and $\psi_\varepsilon^{ex}(x, k)$, respectively, where the quantity k is not replaced by τ_ε. Then, the asymptotic forms of the corresponding quasi-eigenfunctions $\psi_\varepsilon(x)$ have the form $\psi_\varepsilon^{in,ex}(x, \tau_\varepsilon)$. By definition, the coefficients of the series $\psi_\varepsilon^{in,ex}(x, k)$ are analytic in a certain neighbourhood of zero, satisfy the Neumann zeroth boundary condition on $\Gamma_0 \setminus 0$, are the solutions of the Helmholtz equation inside Ω and outside Ω, respectively, and for real k the coefficients of the series $\psi_\varepsilon^{ex}(x, k)$ also satisfy the radiation condition (1.2). As $r \to 0$, $k \to 0$ for Green's functions we have for the limiting problems and their derivatives

$$P_i^{(j)}(D_y) G^{in,ex}(x,0,k) = (-1)^i (2\pi)^{-1} P_i^{(j)}(D_x)(r^{-1}\cos kr) + g_{ji}^{in,ex}(x,k), \quad j \geq 1,$$

$$G^{ex}(x,0,k) = (2\pi)^{-1} r^{-1} \cos kr + g_{00}^{ex}(x,k) \tag{2.2}$$

$$G^{in}(x,0,k) = (2\pi)^{-1} r^{-1} \cos kr - k^{-2} \psi^2 + g_{00}^{in}(x,k)$$

$$\operatorname{Im} G^{in}(x,0,k) \equiv 0, \quad k \in \mathbf{R}, \quad \operatorname{Im} G^{ex}(0,0,k) \sim \operatorname{Re} k\sigma, \quad k \to 0 \tag{2.3}$$

where the functions $g_\varepsilon^{in,ex}(x, k)$ are infinitely differentiable with respect to the variable x and are analytic with respect to the variable k. It is shown in [5] that the last equation holds.

In sums $U(x, \varepsilon)$ of the form $\psi_\varepsilon^{in,ex}(x, \tau_\varepsilon)$ we will define the operator K_q as follows [6, 11]. We will expand the coefficients of the series $U(x, \varepsilon)$ in series as $r \to 0$ and change to the variable $\xi = x\varepsilon^{-2}$. In the double series obtained we take the sum of terms of the form $\varepsilon^j \phi(\xi)$ with $j \leq q$, which we will also represent by $K_q(U(x, \varepsilon))$.

We will call the two asymptotic power series $V^{+,-}(\xi)$ conjugate if their sum is a polynomial. Suppose the functions τ_ε and the series $\psi_\varepsilon^{in,ex}(x, k)$ have asymptotic forms defined by (1.5) and the first and third formulae of (1.6), while the coefficients τ_j, $R_0^{(0)}$ and the differential polynomials $P_i^{(j)}(D_y)$ are arbitrary, but $\tau_1 \neq 0$, $\tau_2 = P_j^{(2j+1)}(D_y) = 0$. Then, it follows from the representations (2.2) that for any integer $N \geqslant 0$

$$K_N(\psi_\varepsilon^{\text{in,ex}}(x,\tau_\varepsilon)) = \sum_{i=0}^{N} \varepsilon^i V_i^{\text{in,ex}}(\xi)$$

the series $V_i^{\text{in,ex}}(\xi)$ are conjugate, and are formal asymptotic solutions of boundary-value problems (2.1) as $\rho = |\xi| \to \infty$, where the functions $\upsilon_i(\xi)$ are replaced by $V_i^{\text{in,ex}}(\xi)$, and for these the following representations hold

$$V_0^{\text{in}}(\xi) = R_0^{(0)}\psi^2 - (2\pi)^{-1}\tau_1^2(R_0^{(0)}\rho^{-1} + \Sigma_0)$$
$$V_0^{\text{ex}}(\xi) = (2\pi)^{-1}\tau_1^2(R_0^{(0)}\rho^{-1} + \Sigma_0) \tag{2.4}$$

$$V_1^{\text{in}}(\xi) = V_1^{\text{ex}}(\xi) \equiv 0$$
$$V_j^{\text{in,ex}}(\xi) = \tilde{V}_j^{\text{in,ex}}(\xi) + 2\tau_1^{-1}\tau_{j+1}(V_0^{\text{in,ex}}(\xi) - \tilde{V}_0^{\text{in,ex}}) \mp (2\pi)^{-1}\tau_1^2\Sigma_j, \quad j \geqslant 2, \tag{2.5}$$

$$\tilde{V}_0^{\text{in}} = R_0^{(0)}\psi^2, \quad \tilde{V}_0^{\text{ex}} = 0$$
$$\Sigma_q = \sum_{i=1}^{\infty} (-1)^i P_i^{(2i+q)}(D_\xi)\rho^{-1} \tag{2.6}$$

where $j \geqslant 2$, $\tilde{V}_0^{\text{in}} = R_0^{(0)}\psi^2$, $\tilde{V}_0^{\text{ex}} = 0$, while the series $\tilde{V}_j^{\text{in,ex}}(\xi)$ are independent of τ_{q+1}, $P_i^{(2i+q)}(D_y)$ when $q \geqslant j$. If, moreover, $\operatorname{Im}\tau_1 = \operatorname{Im} R_0^{(0)} = 0$, then

$$\tilde{V}_2^{\text{in,ex}}(\xi) = \mp\tau_1^2 R_0^{(0)} g_{00}^{\text{in,ex}}(0,0), \quad \operatorname{Im}\tilde{V}_3^{\text{in}}(\xi) \equiv 0, \quad \operatorname{Im}\tilde{V}_3^{\text{ex}}(\xi) = \tau_1^3 R_0^{(0)}\sigma \tag{2.7}$$

by virtue of (2.2) and (2.3).

Hence, the problem of matching the series (1.6) has been reduced to finding solutions $\upsilon_j(\xi) \in W_{2,\text{loc}}^1(\mathbf{R}^3 \setminus \bar{\gamma})$ of boundary-value problems (2.1) which have asymptotic forms as $\rho \to \infty$ which are identical with the series $V_j^{\text{in,ex}}(\xi)$ when $\xi_3 \gtrless 0$. This agreement will be achieved by choosing the constants τ_{j+1} and the polynomials $P_i^{(2i+j)}(D_y)$.

Since $-k^2R_0^{(0)}G^{\text{in}}(x, 0, k) \to R_0^{(0)}\psi^2$, $k^2R_0^{(0)}G^{\text{ex}}(x, 0, k) \to 0$ as $k \to 0$, by virtue of (2.2), while the function $\Psi_\varepsilon \to \psi$ inside Ω, and $\Psi_\varepsilon \to 0$ outside Ω as $\varepsilon \to 0$, then, by choosing $R_0^{(0)}$ in accordance with the first expression in (1.6), we obtain

$$-\tau_\varepsilon^2 R_0^{(0)} G^{\text{in}}(x,0,\tau_\varepsilon) \to \psi, \quad \tau_\varepsilon^2 R_0^{(0)} G^{\text{ex}}(x,0,\tau_\varepsilon) \to 0$$

Boundary-value problem (2.1) for $\upsilon_0(\xi)$ has the form

$$\Delta_\xi \upsilon_0 = 0, \quad \xi \notin \bar{\gamma}, \quad \partial\upsilon_0 / \partial\xi_3 = 0, \quad \xi \in \gamma \tag{2.8}$$

By definition, the function $Y(\xi)$, harmonic outside $\bar{\gamma}$, has the following symptotic forms as $\rho \to \infty$

$$Y(\xi) = \begin{cases} 1 - \tfrac{1}{2}X(\xi), & \xi_3 \geqslant 0 \\ \tfrac{1}{2}X(\xi), & \xi_3 \leqslant 0 \end{cases}, \quad X(\xi) = \frac{1}{2}c_\omega\rho^{-1} + \sum_{i=1}^{\infty} Z_i(\xi)\rho^{-1-2i}$$

and its derivative with respect to ξ_3 is zero on γ; $Z_i(\xi)$ are uniform harmonic polynomials of degree i. We put $\upsilon_0(\xi) = R_0^{(0)}\psi^2Y(\xi)$. The function $\upsilon_0 \in W_{2,\text{loc}}^1(\mathbf{R}^3 \setminus \bar{\gamma})$ is a solution of boundary-value problem (2.8), and by virtue of (2.4) the following equations hold as $\rho \to \infty$

$$\upsilon_0(\xi) - V_0^{\text{in,ex}}(\xi) = R_0^{(0)}((2\pi)^{-1}\tau_1^2 - \tfrac{1}{2}\psi^2c_\omega)\rho^{-1} + \sum_{i=1}^{\infty} ((2\pi)^{-1}\tau_1^2(-1)^i P_i^{(2i)}(D_\xi)\rho^{-1} - Z_i(\xi)\rho^{-1-2i}), \quad \xi_3 \gtrless 0$$

Equating the coefficient of ρ^{-1} on the right-hand side of the last equation to zero, we obtain the value of (1.5) for τ_1. Equating the coefficients of the remaining powers of ρ to zero we obtain $P_i^{(2i)}(D_y)$. Note that by determining $P_1^{(2)}(D_y)$ we finally find $R_1^{(2)}(D_y)$. Hence, we have carried out the "zeroth" matching step.

The further proof is carried out by induction. At the jth step, by equating the asymptotic forms at infinity of the solution $\upsilon_j(\xi)$ of problem (2.1) to the series (2.6), we determine the coefficients τ_{j+1} and the polynomials $P_i^{(2i+j)}(D_y)$, and consequently the polynomials $R_{[j/2]+1}^{(j+2)}(D_y)$ also. The equations

$$\tau_2 = \upsilon_1(\xi) = 0, \quad \upsilon_2(\xi) = \tau_1^2 R_0^{(0)} (g_{00}^{\text{ex}}(0,0)Y(\xi_1,\xi_2,-\xi_3) - g_{00}^{\text{in}}(0,0)Y(\xi))R_0^{(0)}$$

and (1.5) for τ_3 follow from (2.5)–(2.7).

Equation (1.5) also holds for $\operatorname{Im}\tau_4$.

In fact, by taking the imaginary part of (2.1) we obtain the boundary-value problem

$$\Delta_\xi \operatorname{Im}\upsilon_3 = 0, \quad \xi \notin \bar{\gamma}, \quad \partial \operatorname{Im}\upsilon_3/\partial\xi_3 = 0, \quad \xi \in \gamma \tag{2.9}$$

In view of (2.6) and (2.7) as $\rho \to \infty$

$$\operatorname{Im}\upsilon_3(\xi) = O(\rho^{-1}), \quad \xi_3 \geq 0, \quad \operatorname{Im}\upsilon_3(\xi) = \tau_1^3 R_0^{(0)}\sigma + O(\rho^{-1}), \quad \xi_3 \leq 0 \tag{2.10}$$

The function $\operatorname{Im}\upsilon_3(\xi) = \tau_1^3 R_0^{(0)}\sigma Y(\xi_1, \xi_2, -\xi_3)$ is obviously a solution of boundary-value problem (2.9), and as $\rho \to \infty$ has the asymptotic forms (2.10). Equating the coefficients of ρ^{-1} in the series $\operatorname{Im}V_3^{\text{in}}(\xi)$ and in the asymptotic form of the function $\operatorname{Im}\upsilon_3(\xi)$ when $\xi_3 > 0$, we can determine $\operatorname{Im}\tau_4$.

As a result of this procedure for any integer $N \geq 0$ as $\rho \to \infty$ we have

$$K_N(\psi_\varepsilon^{\text{in,ex}}(x,\tau_\varepsilon)) = \sum_{i=0}^{N} \varepsilon^i \upsilon_i(\xi), \quad \xi_3 \gtrless 0, \quad \upsilon_i(\xi) = O(\rho^{[i/2]}) \tag{2.11}$$

We extend the partial sums

$$\psi_{\varepsilon,N}^{\text{in,ex}}(x,k) = \mp k^2 \sum_{j=0}^{N} \varepsilon^j R_{[j/2]}^{(j)}(D_y) G^{\text{in,ex}}(x,0,k)$$

to zero outside $\bar{\Omega}$ and inside Ω, respectively, and we denote by $\upsilon_{\varepsilon,N}(\xi)$ the partial sum of the second series of (1.6) and by $\chi(t)$ the infinitely differentiable cutoff function, which is identically equal to zero when $t<1$ and unity when $t>2$. By definition, the function

$$\psi_{\varepsilon,N}(x,k) = \chi(r\varepsilon^{-1})(\psi_{\varepsilon,N}^{\text{in}}(x,k) + \psi_{\varepsilon,N}^{\text{ex}}(x,k)) + (1-\chi(r\varepsilon^{-1}))\upsilon_{\varepsilon,N}(\xi)$$

belongs to $W_{2,\text{loc}}^1(\mathbf{R}^3 \setminus \bar{\Gamma}_\varepsilon)$, is analytic with respect to k in a certain neighbourhood of zero, and is a solution of the boundary-value problem

$$(\Delta + k^2)\psi_{\varepsilon,N} = f_{\varepsilon,N}, \quad x \in \mathbf{R}^3 \setminus \bar{\Gamma}_\varepsilon, \quad \frac{\partial\psi_{\varepsilon,N}}{\partial \mathbf{n}} = 0, \quad x \in \Gamma_\varepsilon$$

where the function $f_{\varepsilon,N}(x, k)$ is analytic with respect to k, and $\operatorname{supp} f_{\varepsilon,N}(x, k) \subset S(2\varepsilon)$. The function $\psi_{\varepsilon,N}(x, k)$ satisfies the radiation condition (1.2) for real k. In view of (2.1) and (2.11) the norm of $f_{\varepsilon,N}(x, \tau_\varepsilon)$ in $L_2(\mathbf{R}^3)$ is of the order of ε^{N_1}, where N_1 increases without limit as $N \to \infty$ (see, for example, [16]). The asymptotic expansions of τ_ε, $\psi_\varepsilon(x)$ are constructed. The justification for this follows from [6, 7].

The flattening of the resonator in the neighbourhood of the aperture has no effect on the values of τ_1, $\operatorname{Im}\tau_{2,3,4}$, $R_0^{(0)}$ and on representation (1.8) and (1.10), which is connected with the asymptotic forms $G^{\text{in,ex}}(x, 0, k)$ [12, 13]. When Ω is a sphere, and the projection of the aperture ω_ε on to the tangential plane is a circle, the value of τ_1 is identical with the value of this quantity obtained by non-rigorous Rayleigh methods.

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